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On the geometry of a class of N-qubit entanglement monotones

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Abstract

A family of N-qubit entanglement monotones invariant under stochastic local operations and classical communication (SLOCC) is defined. This class of entanglement monotones includes the well-known examples of the concurrence, the 3-tangle and some of the four-, five- and N-qubit SLOCC invariants introduced recently. The construction of these invariants is based on bipartite partitions of the Hilbert space in the form $\mathbf{C}^{2^N} \simeq \mathbf{C}^L \otimes \mathbf{C}^l$ with $L = 2^{N-n} \geqslant l = 2^n$. Such partitions can be given a nice geometrical interpretation in terms of Grassmannians $Gr(L, l)$ of l-planes in $\mathbf{C}^L$ that can be realized as the zero locus of quadratic polynomials in the complex projective space of suitable dimension via the Plücker embedding. The invariants are neatly expressed in terms of the Plücker coordinates of the Grassmannians.

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1. Introduction

Since the advent of quantum information science [1] which regards entanglement as a resource, it has become of fundamental importance to characterize different classes of entanglement via the use of suitable entanglement measures. Though there are a number of very useful and spectacular results [2–6] for quantifying the amount of entanglement present in pure and mixed states of multipartite systems, the subject is still at its infancy. For pure states, for example, we know that it is unlikely that the complete classification of N-qubit states will ever be found [7] due to formidable computational difficulties. Under such conditions, it seems reasonable to try to find a characteristic subclass of N-qubit entanglement that can be described in a unified way. In this paper, we attempt a modest step towards the identification of such a class which provides a way of understanding N-qubit entanglement in geometric terms.

The use of geometric ideas in understanding entanglement has already been used in a number of papers [8–12]. In particular, it was observed [10, 11] that two-qubit entangled states and a special class of three-qubit entangled states can be described by certain maps that are entanglement sensitive. These maps enable a geometric description of entanglement in terms of fibre bundles. Fibre bundles are spaces which locally look like the product of two spaces, the base and the fibre, globally, however they can exhibit a nontrivial twisted structure. In this picture, this twisting of the bundle accounts for some portion of quantum entanglement. For two qubits, these ideas were elaborated [13] using the correspondence between fibre bundles and the language of gauge fields. The essence of this approach was to provide a description of entanglement by regarding the local unitary (LU) transformations corresponding to a *fixed* subsystem as gauge degrees of freedom. In our recent paper [14], we have generalized this approach to describe the interesting geometry of three-qubit entanglement. For this purpose, we have taken into account the more general class of transformations corresponding to stochastic local operations and classical communication (SLOCC). Using twistor methods we have shown that the relevant fibration in this case is a one over the Grassmannian $Gr(4, 2)$ of 2-planes in $\mathbf{C}^4$ with the gauge group being the SLOCC transformations of an arbitrarily chosen qubit, i.e. $GL(2, \mathbf{C})$. For every three-qubit state, we have associated a *pair* of planes in $\mathbf{C}^4$, or equivalently a *pair* of lines in the complex projective space $\mathbf{CP}^3$. In this picture, entanglement can be described by the intersection properties of a pair of lines in $\mathbf{CP}^3$. Unlike the one in [11], this method turned out to be capable of characterizing geometrically all the entanglement classes introduced in [15]. For example, the two inequivalent classes of genuine three-party entanglement, namely the GHZ and W classes, correspond to the geometric situation of a pair of nonintersecting lines or lines intersecting in a point, respectively.

The aim of the present paper is to generalize these geometric ideas for multiqubit systems. We will see that for an interesting subclass of N-qubit entanglement such a generalization can indeed be done. The starting point of our investigations is a recent paper of Emary [16] introducing a class of entanglement monotones based on bipartite partitions of multiqubit systems. By reformulating and generalizing the results of [16], we are naturally led to a class of SLOCC entanglement monotones giving back the well-known examples of the concurrence [6], the 3-tangle [17] and some of the four- [7], five- [18] and N-qubit [3] invariants introduced recently. Moreover, these invariants can be rewritten in a nice and instructive form of geometric significance. In fact, these invariants are the natural ones associated with higher dimensional Grassmannians of l-planes that can be embedded in a complex projective space of suitable dimension. This observation leads us to the interesting possibility of understanding entanglement in terms of the intersection properties of projective subspaces of a complex projective space of suitable dimension. This approach being interesting and useful in its own right also shows a nice connection with twistor theory [14, 19].

2. A bipartite class of entanglement monotones

As a starting point, we reformulate the results of [16] in a geometric fashion convenient for our purposes. Let us consider an arbitrary N-qubit pure normalized state $|\Psi\rangle \in \mathbf{C}^{2^N}$

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N=0}^{1} C_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle, \tag{1}$$

where the states $|i_1 i_2 \cdots i_N\rangle \equiv |i_1\rangle \otimes |i_2\rangle \cdots \otimes |i_N\rangle$ correspond to the computational base of our N-qubit state. Let us single out n qubits such that $L \equiv 2^{N-n} \geqslant l \equiv 2^n$. For convenience, we chose these qubits to be the *last* n ones from the list $i_1 i_2 \cdots i_N$, i.e. we have $i_1 i_2 \cdots i_{N-n} i_{N-n+1} \cdots i_N$. Let us now construct the $L \times l$ matrix $Z_{\alpha a}, \alpha = 0, 1, \ldots, L-1$, $a = 0, 1, \ldots, l-1$, of $2^N = L \times l$ complex entries to be just $C_{i_1 i_2 \cdots i_N}$ arranged according to this partition. This means that the first $N-n$ terms of the binary string $i_1 i_2 \cdots i_N$ written in decimal form are represented by α (rows) while the remaining n terms in decimal form are represented by the letter a (columns). Since according to our assumption $N - n \geqslant n$, the matrix $Z_{\alpha a}$ is of rectangular shape with the number of rows greater than or equal to the number of columns.

Let us assume now that the *columns* $Z_{\alpha 0}, Z_{\alpha 1}, \ldots, Z_{\alpha l-1} \equiv \mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{l-1}$ considered as *unnormalized* vectors in $\mathbf{C}^L$ are linearly independent. Then, the matrix $(Z^\dagger Z)_{ab}$ (the reduced density matrix of the last n qubits) is of maximal rank. Hence, the assumption of linear independence for *all* bipartite partitions is equivalent to the one that $|\Psi\rangle$ reinterpreted as the state of a *bipartite* system in $\mathbf{C}^{N-n} \otimes \mathbf{C}^n$ for all $N - n \geqslant n$ is totally entangled [20].

Our unnormalized linearly independent vectors $\mathbf{Z}_0, \mathbf{Z}_1, \ldots, \mathbf{Z}_{l-1}$ span an l-plane in $\mathbf{C}^L$. The set of l-planes in $\mathbf{C}^L$ forms an $(L - l \times l)$-dimensional complex manifold, the Grassmannian $Gr(L, l)$. There are a number of ways to introduce complex coordinates for this manifold. First, the entries of the $L \times l$ matrix define the so-called homogeneous or Stiefel coordinates. Their number is greater than the (complex) dimension of the manifold. This redundancy in the homogeneous coordinates has its origin in the fact that any linear combination of the vectors $\mathbf{Z}_a$, $a = 0, 1, \ldots, l - 1$, spans the same l-plane. Equivalently, the transformation $Z \mapsto ZS$ where $S \in GL(l, \mathbf{C})$ (the set of invertible $l \times l$ matrices with complex entries) can be regarded as a gauge degree of freedom. It merely amounts to a redefinition of the vectors spanning the l-plane in question. It can be shown [21] that $S(L, l)$ the set of complex $L \times l$ matrices $Z_{\alpha a}$ of full rank forms a fibre bundle over the Grassmannian $Gr(L, l)$ with gauge group, i.e. we have $Gr(L, l) = S(L, l)/GL(l, \mathbf{C})$.

Another way of defining homogeneous coordinates for $Gr(L, l)$ is to use the so-called Plücker coordinates. By definition, the Plücker coordinate $P_{\alpha_0\alpha_1\cdots\alpha_{l-1}}$ of the l-plane defined by Z is just the maximal minor of $Z_{\alpha a}$ formed by using the rows singled out by the l fixed values $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$. It is obvious that if we make the transformation $Z \mapsto ZS$ with $S \in GL(l, \mathbf{C})$, the Plücker coordinates transform as $P_{\alpha_0\alpha_1\cdots\alpha_{l-1}} \mapsto \mathrm{Det}(S) P_{\alpha_0\alpha_1\cdots\alpha_{l-1}}$. The number of such coordinates is $\binom{L}{l}$ which is greater than the dimension of the Grassmannian $Gr(L, l)$, this means that the Plücker coordinates are not independent. They are subject to special relations called the Plücker relations.

In order to illustrate these abstract concepts, let us consider the example of a three-qubit system $N = 3$. The state of the system can then be written in the form

$$|\Psi\rangle = \sum_{i_1, i_2, i_3 = 0}^{1} C_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle. \tag{2}$$

Let us choose $n = 1$ corresponding to the *last* qubit, then we have $L = 4$ $(\alpha = 0, 1, 2, 3)$ and $l = 2$ $(a = 0, 1)$, hence

$$\mathbf{Z}_0 \equiv \begin{pmatrix} Z_{00} \\ Z_{10} \\ Z_{20} \\ Z_{30} \end{pmatrix} = \begin{pmatrix} C_{000} \\ C_{010} \\ C_{100} \\ C_{110} \end{pmatrix}, \tag{3}$$

$$\mathbf{Z}_1 \equiv \begin{pmatrix} Z_{01} \\ Z_{11} \\ Z_{21} \\ Z_{31} \end{pmatrix} = \begin{pmatrix} C_{001} \\ C_{011} \\ C_{101} \\ C_{111} \end{pmatrix}. \tag{4}$$

Now, the Plücker coordinates are the maximal minors of the 4×2 matrix $Z_{\alpha a}$ formed by the columns above. Let us choose arbitrarily two values $\alpha_0 = \alpha$ and $\alpha_1 = \beta$, then the Plücker coordinates are

$$P_{\alpha\beta} = Z_{\alpha 0} Z_{\beta 1} - Z_{\beta 0} Z_{\alpha 1}. \tag{5}$$

The number of such coordinates is 6 which is greater than the complex dimension of $Gr(4, 2)$ which is 4.

Clearly, under a $GL(2, \mathbf{C})$ transformation

$$\begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \\ Z_{20} & Z_{21} \\ Z_{30} & Z_{31} \end{pmatrix} \mapsto \begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \\ Z_{20} & Z_{21} \\ Z_{30} & Z_{31} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{6}$$

these coordinates transform as

$$P_{\alpha\beta} \mapsto (AD - BC) P_{\alpha\beta}. \tag{7}$$

Hence, although the number of Plücker coordinates is greater than the complex dimension of $Gr(4, 2)$, we see from the above equation that these coordinates are defined merely projectively, i.e. up to a nonzero complex number, hence only their ratios count as coordinates. The number of ratios is 5, moreover one can check that the quadratic Plücker relation

$$P_{01} P_{23} - P_{02} P_{13} + P_{03} P_{12} = 0 \tag{8}$$

holds which reduces the number of independent complex coordinates to 4, the complex dimension of $Gr(4, 2)$.

As we have seen, the Plücker coordinates are defined up to a common scalar factor. Since these coordinates are defined merely projectively, we should be able to embed $Gr(L, l)$ into the complex projective space $\mathbf{CP}^D$ with $D = \binom{L}{l} - 1$. Such embedding really exists, it is the Plücker embedding

$$Gr(L, l) \hookrightarrow \mathbf{CP}^{\binom{L}{l}-1} = \mathbf{P}\left(\bigwedge^{l} \mathbf{C}^L\right) \tag{9}$$

associating with the vectors $\mathbf{Z}_a$, $a = 0, \ldots, l - 1$, spanning the l-plane in question the *separable* l-vector $\mathbf{Z}_0 \wedge \mathbf{Z}_1 \wedge \cdots \wedge \mathbf{Z}_{l-1}$ in the l-fold antisymmetric tensor product of $\mathbf{C}^L$ with itself. For the three-qubit case, the Plücker embedding associates with the 2-plane determined by the vectors $\mathbf{Z}_0$ and $\mathbf{Z}_1$ the separable *bivector* $\mathbf{Z}_0 \wedge \mathbf{Z}_1$. Writing out this bivector as an antisymmetric matrix, we get equation (5). Hence, we can alternatively regard the Plücker coordinates as separable l-vectors or as totally antisymmetric matrices with l indices, satisfying additional constraints (Plücker relations). In the language of l-vectors, the transformation property of Plücker coordinates is

$$\mathbf{Z}_0 \wedge \cdots \wedge \mathbf{Z}_{l-1} \mapsto (\mathrm{Det}\ S) \mathbf{Z}_0 \wedge \cdots \wedge \mathbf{Z}_{l-1}, \tag{10}$$

where $S \in GL(l, \mathbf{C})$ is the usual $l \times l$ matrix acting on our $L \times l$ matrix Z. Clearly, equation (7) is just a special case of (10).

After illustrating our use of the Plücker coordinates, let us use them to express the entanglement monotones of [16] in a simpler form. For this, following [16] let us introduce the operator $\mathrm{d}x_\alpha$, which assigns to vectors $\{\mathbf{Z}\} \equiv \{\mathbf{Z}_0, \ldots, \mathbf{Z}_{l-1}\}$ their αth component, i.e. $\mathrm{d}x_\alpha(\mathbf{Z}_a) = Z_{\alpha a}$, and combines them in the wedge product defined as

$$\bigwedge_{a=0}^{l-1} \mathrm{d}x_{\alpha_a}(\{\mathbf{Z}\}) = \mathrm{Det}(\mathrm{d}x_{\alpha_a}(\mathbf{Z}_b))_{a,b=0,\ldots,l-1}. \tag{11}$$

Clearly, this quantity is just the maximal minor of Z labelled by the rows α_a, $a = 0, \ldots, l-1$, i.e. the Plücker coordinate $P_{\alpha_1\cdots\alpha_{l-1}}$. In this notation, the entanglement monotones

$$D_n^{(k_1,\ldots,k_n)} \equiv l^2 \left(\sum_{\alpha_0<\cdots<\alpha_{l-1}=0}^{L-1} \left| \bigwedge_{a=0}^{l-1} \mathrm{d}x_{\alpha_a}(\{\mathbf{Z}\}) \right|^2 \right)^{2/l} \tag{12}$$

of [16] take the following instructive form:

$$D_n^{(k_1,\ldots,k_n)} \equiv \frac{l^2}{l!} \left(\sum_{\alpha_0,\ldots,\alpha_{l-1}=0}^{L-1} \left| P_{\alpha_1\cdots\alpha_{l-1}} \right|^2 \right)^{2/l}. \tag{13}$$

Note that here we have introduced the general notation $(k_1, \ldots, k_n)$ of [16] to identify the location of n qubits. In our simplified case, $(k_1, \ldots, k_n) = (N-n+1, \ldots, N)$, i.e. we have placed n qubits to the end of the N-qubit string. Clearly, our considerations can be repeated for any partition with n qubit locations labelled as $(k_1, \ldots, k_n)$ and a suitable adjustment for the definition of the Plücker coordinates for this case. It should be obvious that for each such partition with fixed L and l, we have a *different* bundle of the form $S(L,l)/GL(l,\mathbf{C})$. For a given n, we have $\binom{N}{n}$ such entanglement monotones associated with these bundles, except for $n = N/2$ where we have half of this number. The important property of the quantities $D_n^{(k_1,\ldots,k_n)}$ is that they are invariant under local unitary (LU) transformations of the qubits [16]. Moreover, writing $P_{\alpha_1\cdots\alpha_{l-1}} = (\mathbf{Z}_0 \wedge \cdots \wedge \mathbf{Z}_{l-1})_{\alpha_1\cdots\alpha_{l-1}}$ and using equation (10) in (13), we see that they are also invariant under the more general transformations of $U(l)$ acting on the l-qubit Hilbert subspace. Note that the quantities $D_n^{(k_1,\ldots,k_n)}$ are not necessarily independent.

3. SLOCC entanglement monotones

In order to motivate our generalization of the LU entanglement monotones (13) to SLOCC entanglement monotones, we turn once again to the three-qubit case. Let us single out the last qubit to be the one characterizing the partition. Then, we can write the antisymmetric matrix of Plücker coordinates in the form $\mathbf{P} = \mathbf{Z}_0 \wedge \mathbf{Z}_1$, i.e. as a separable bivector (see equations (3)–(5)). Then, we have $l = 2$ and $L = 4$ and the entanglement monotone $D_1^{(3)}$ can be written in the form

$$D_1^{(3)} = 2 \sum_{\alpha\beta=0}^{3} |P_{\alpha\beta}|^2 = 4\,\mathrm{Det} \begin{pmatrix} \langle \mathbf{Z}_0|\mathbf{Z}_0\rangle & \langle \mathbf{Z}_0|\mathbf{Z}_1\rangle \\ \langle \mathbf{Z}_1|\mathbf{Z}_0\rangle & \langle \mathbf{Z}_1|\mathbf{Z}_1\rangle \end{pmatrix}, \tag{14}$$

where $\langle \mathbf{Z}_a|\mathbf{Z}_b\rangle \equiv \sum_{\alpha=0}^{3} \overline{Z}_{\alpha a} Z_{\alpha b}$, where the overbar denotes the complex conjugation. As it is well known [17, 22, 23], $D_1^{(3)} = \tau_{(12)3} = 4\,\mathrm{Det}\,\rho_3 = 2\big(1 - \mathrm{Tr}\,\rho_3^2\big)$ which is the linear entropy of the third qubit. Repeating the same construction with the first and then the second qubit, one gets the monotones $D_1^{(1)}$ and $D_1^{(2)}$ related to the linear entropies of these qubits. The quantity $Q_1 = \frac{1}{3}\big(D_1^{(1)} + D_1^{(2)} + D_1^{(3)}\big)$ is the permutation invariant used in [22, 23].

Let us now introduce a bilinear form $g : \mathbf{C}^4 \times \mathbf{C}^4 \to \mathbf{C}$ such that for two vectors $\mathbf{A}, \mathbf{B} \in \mathbf{C}^4$ we have

$$(\mathbf{A}, \mathbf{B}) \mapsto g(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B} = g_{\alpha\beta} A^{\alpha} B^{\beta} = A_{\alpha} B^{\alpha}, \tag{15}$$

where

$$g_{\alpha\beta} = g_{i_1 i_2, j_1 j_2} = \varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2}, \tag{16}$$

or explicitly

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{17}$$

$\alpha, \beta = 0, 1, 2, 3$ and summation for repeated indices is understood. Clearly, $\overline{A \cdot B} = -\langle A|\tilde{B}\rangle$, where the right-hand side is expressed via the spin flip operation of [6], i.e. $|\tilde{B}\rangle = \sigma_2 \otimes \sigma_2 |\overline{B}\rangle$.

Let us now define the quantity similar to the one in (14)

$$E_1^{(3)} \equiv 2|P_{\alpha\beta} P^{\alpha\beta}| = 4 \left| \mathrm{Det} \begin{pmatrix} \mathbf{Z}_0 \cdot \mathbf{Z}_0 & \mathbf{Z}_0 \cdot \mathbf{Z}_1 \\ \mathbf{Z}_1 \cdot \mathbf{Z}_0 & \mathbf{Z}_1 \cdot \mathbf{Z}_1 \end{pmatrix} \right|. \tag{18}$$

Note the crucial changes we have made, namely we have taken the modulus of the sum, and the sum was understood with respect to the metric (16). Since $M\varepsilon M^t = \varepsilon$ with $M \in SL(2, \mathbf{C})$, this sum with respect to g is invariant under $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$, i.e. of determinant 1 SLOCC transformations acting on the first and second qubits, respectively. Moreover, (7) transformation property shows that the Plücker coordinates are invariant under the remaining $SL(2, \mathbf{C})$ transformation of the third qubit. Hence, $E_1^{(3)}$ is an $SL(2, \mathbf{C})^{\otimes 3}$ invariant which can be shown [14] to be the 3-tangle τ_{123} [17] which is also an entanglement monotone [15]. Moreover, it is easy to check that the invariants $E_1^{(1)}$ and $E_1^{(2)}$ defined similarly are equal to $E_1^{(3)}$, reflecting the permutation invariance of the 3-tangle.

Having gained some insight into the structure of three-qubit invariants, now we turn to our generalization of SLOCC entanglement monotones. (In the following by SLOCC transformations we mean the group $SL(2, \mathbf{C})^{\otimes N}$.) The monotones we wish to propose are of the form

$$E_n^{(k_1,\ldots,k_n)} \equiv \frac{l^2}{l!} \left| P_{\alpha_0\cdots\alpha_{l-1}} P^{\alpha_0\cdots\alpha_{l-1}} \right|^{2/l}, \tag{19}$$

where summation is now understood with respect to the $SL(2, \mathbf{C})^{\otimes(N-n)}$ invariant bilinear form with matrix

$$g_{\alpha\beta} = \varepsilon_{i_0 j_0} \otimes \cdots \otimes \varepsilon_{i_{N-n-1} j_{N-n-1}}. \tag{20}$$

Hence, the matrix of g is just the $(N-n)$-fold tensor product of the fundamental $SL(2, \mathbf{C})$ invariant tensor ε. An alternative formula using the l linearly independent vectors spanning the l-plane in question is

$$E_n^{(\{k\})} \equiv l^2 \left| \mathrm{Det} \begin{pmatrix} \mathbf{Z}_0 \cdot \mathbf{Z}_0 & \cdots & \mathbf{Z}_0 \cdot \mathbf{Z}_{l-1} \\ \cdots & \cdots & \cdots \\ \mathbf{Z}_{l-1} \cdot \mathbf{Z}_0 & \cdots & \mathbf{Z}_{l-1} \cdot \mathbf{Z}_{l-1} \end{pmatrix} \right|^{2/l}. \tag{21}$$

In the following, we adopt the convention of regarding the bilinear form g to be fundamental, i.e. we consider the pair $(\mathbf{C}^L, g)$ meaning that $\mathbf{C}^L$ is equipped with the extra structure defined by g. Note that for $N-n$ even the matrix g is symmetric and for $N-n$ odd it is antisymmetric. For $N-n$ odd, g defines a simplectic structure on $\mathbf{C}^L$.

The $SL(2, \mathbf{C})^{\otimes(N-n)}$ invariance of the quantities $E_n^{(\{k\})}$ ($\{k\} \equiv (k_1, k_2, \ldots, k_n)$) follows from the invariance of the bilinear form and the $SL(2, \mathbf{C})^{\otimes n}$ invariance follows from the (10) transformation formula of the Plücker coordinates used for the subgroup $SL(2, \mathbf{C})^{\otimes n} \subset SL(l, \mathbf{C})$. Hence, $E_n^{(k)}$ are invariant under the full group of determinant 1 SLOCC transformations, i.e. $SL(2, \mathbf{C})^{\otimes N}$.

The other important property of the quantities $E_n^{(\{k\})}$ is that they are entanglement monotones, meaning that on average they are non-increasing under the action of any local protocol. Now any local protocol can be decomposed into POVM (positive operator valued measures) acting on a single qubit. Since any POVM can further be decomposed into a sequence of two-outcome POVMs, it is enough to demonstrate the non-increasing property of $E_n^{(\{k\})}$ under two-outcome POVMs. The proof that this property is indeed satisfied is simply a slightly modified rerun of the standard arguments that can be found in [15, 16, 24]. The choice of the power $2/l$ in definition (19) makes $E_n^{(\{k\})}$ to transform under local POVMs in the same way as the concurrence squared and the 3-tangle do [16].

4. Examples

4.1. Two and three qubits

As our first example, it is easy to show that in the case of two qubits ($N = 2, n = 1, L = l = 2$), our entanglement monotones give back the usual definition of the concurrence squared. Indeed, in this case $Z_{\alpha a} = C_{\alpha a}(\alpha, a = 0, 1)$, hence we have a 2×2 matrix $Z = C$ with linearly independent columns. Two linearly independent vectors in $\mathbf{C}^2$ define the trivial Grassmannian $Gr(2, 2)$ which is just a point. For the monotone $E_1^{(2)}$, we have the formula

$$E_1^{(2)} = 4|\mathrm{Det}(\mathbf{Z}_a \cdot \mathbf{Z}_b)| = 4|\mathrm{Det}(Z^t g Z)| = 4|\mathrm{Det}\, C|^2, \tag{22}$$

which is just the concurrence squared. Clearly, $E_1^{(1)} = E_1^{(2)}$.

For the three-qubit case, we have already shown that $E_1^{(1)} = E_1^{(2)} = E_1^{(3)} = \tau_{123}$ with τ_{123} being the 3-tangle. Moreover, from equation (18) we see that $E_1^{(3)}$ is just four times the magnitude of the discriminant of $\mathrm{Det}\big(xC_{i_1i_20} + yC_{i_1i_21}\big) = 0$, i.e. a binary form of degree 2 in the complex variables x and y. According to the method of Schläfli [21], this discriminant is just the hyperdeterminant $D(C)$ of $C_{i_1i_2i_3}$.

4.2. Four qubits

As our first nontrivial example, let us consider an arbitrary four-qubit state

$$|\Psi\rangle = \sum_{i_1,i_2,i_3,i_4=0}^{1} C_{i_1i_2i_3i_4}|i_1i_2i_3i_4\rangle. \tag{23}$$

Let us first consider the partition $N - n = 3, n = 1$. In this case $L = 8$ and $l = 2$, hence for each four-qubit state totally entangled for this partition we have a 2-plane in $\mathbf{C}^8$. Geometrically, a four-qubit state of this kind determines a point in the Grassmannian $Gr(8, 2)$, or equivalently a *line* in $\mathbf{CP}^7$. Moreover, the Grassmannian $Gr(8, 2)$ as a manifold of complex dimension 12 can be embedded in $\mathbf{CP}^{27}$ via the Plücker embedding. In this case, we have a 8×2 matrix $Z_{\alpha a}$ with $\alpha = 0, 1, \ldots, 7$ and $a = 0, 1$ consisting of the two columns

$$\mathbf{Z}_0 \equiv \begin{pmatrix} Z_{00} \\ Z_{10} \\ Z_{20} \\ Z_{30} \\ Z_{40} \\ Z_{50} \\ Z_{60} \\ Z_{70} \end{pmatrix} = \begin{pmatrix} C_{0000} \\ C_{0010} \\ C_{0100} \\ C_{0110} \\ C_{1000} \\ C_{1010} \\ C_{1100} \\ C_{1110} \end{pmatrix} = \begin{pmatrix} C_0 \\ C_2 \\ C_4 \\ C_6 \\ C_8 \\ C_{10} \\ C_{12} \\ C_{14} \end{pmatrix}, \tag{24}$$

$$\mathbf{Z}_1 \equiv \begin{pmatrix} Z_{01} \\ Z_{11} \\ Z_{21} \\ Z_{31} \\ Z_{41} \\ Z_{51} \\ Z_{61} \\ Z_{71} \end{pmatrix} = \begin{pmatrix} C_{0001} \\ C_{0011} \\ C_{0101} \\ C_{0111} \\ C_{1001} \\ C_{1011} \\ C_{1101} \\ C_{1111} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_3 \\ C_5 \\ C_7 \\ C_9 \\ C_{11} \\ C_{13} \\ C_{15} \end{pmatrix}, \tag{25}$$

where for later use we have also written out explicitly the four-qubit amplitudes using the decimal labelling. The bilinear form on $\mathbf{C}^8$ is antisymmetric with the explicit form

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Due to the antisymmetry of g, we have $\mathbf{Z}_0 \cdot \mathbf{Z}_0 = \mathbf{Z}_1 \cdot \mathbf{Z}_1 = 0$; hence, for the entanglement monotone $E_4^{(4)}$, we have the formula

$$E_1^{(4)} = 4\,\mathrm{Det}\left|\begin{pmatrix} 0 & \mathbf{Z}_0 \cdot \mathbf{Z}_1 \\ \mathbf{Z}_1 \cdot \mathbf{Z}_0 & 0 \end{pmatrix}\right| = 4|\mathbf{Z}_0 \cdot \mathbf{Z}_1|^2. \tag{27}$$

We can also write this using the decimal labelling of the four-qubit amplitudes as

$$E_1^{(4)} = 4|C_0C_{15} - C_2C_{13} - C_4C_{11} + C_6C_9 - C_8C_7 + C_{10}C_5 + C_{12}C_3 - C_{14}C_1|^2. \tag{28}$$

Hence, $E_1^{(4)} = 4|H|^2$, where H is the $SL(2, \mathbf{C})^{\otimes 4}$ invariant introduced in [7]. Calculating the invariants $E_1^{(1)}$, $E_1^{(2)}$ and $E_1^{(3)}$ by choosing the reduced qubits to be the first, second and third, respectively, a similar calculation shows that they are all equal to $E_1^{(4)}$ in accordance with the permutation invariance of H [7]. Later, when we look at this invariant in a more general context, we will give a simple proof of this fact.

Let us now calculate the invariant E_2^{34}. In this case, we have $N - n = 2$ and $n = 2$, hence $L = l = 4$. In this case, we have four vectors in $\mathbf{C}^4$, hence the Grassmannian $Gr(4, 4)$ being a point is again trivial. One then shows that

$$Z_{\alpha a} = \begin{pmatrix} C_0 & C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 & C_7 \\ C_8 & C_9 & C_{10} & C_{11} \\ C_{12} & C_{13} & C_{14} & C_{15} \end{pmatrix}. \tag{29}$$

Hence, similar to the two-qubit case, we have merely one Plücker coordinate which is just the determinant of the matrix above, then we have

$$E_2^{(34)} = 16|\mathrm{Det}(\mathbf{Z}_a \cdot \mathbf{Z}_b)|^{1/2} = 16|\mathrm{Det}\, Z|, \tag{30}$$

hence $E_2^{(34)} = 16\ \mathrm{Det}|L|$, where L is the $SL(2, \mathbf{C})^{\otimes 4}$ invariant introduced in [7]. We can calculate two more invariants of this kind, namely $E_2^{(24)}$ and $E_2^{(14)}$ (the remaining ones are not independent). A calculation shows that $E_2^{(24)} = 16|M|$ and $E_2^{(14)} = 16|N|$ in the notation of [7]. The $SL(2, \mathbf{C})^{\otimes 4}$ invariants L, M and N are still not independent due to the relation

$L + M + N = 0$. Note also that the same invariants arise from the ones of Emary, namely $D_2^{(34)}$, $D_2^{(24)}$ and $D_2^{(14)}$, due to the fact that in this very special case the number of Plücker coordinates is merely 1, so the sums in (13) and (19) contain merely one term (the sum of magnitudes in this case equals the magnitude of the sum). Moreover, since the Hilbert series for the algebra of $SL(2, \mathbf{C})^{\otimes 4}$ invariants is known [7], it follows that the invariants $E_1^{(4)}$, $E_2^{(34)}$ and $E_2^{(24)}$ are algebraically independent. Moreover, there are four invariants of degrees 2, 4, 4 and 6, generating freely the algebra of SLOCC invariants of a four-qubit system. Our monotones already reproducing three of such fundamental invariants.

4.3. Five qubits

For a five-qubit state

$$|\Psi\rangle = \sum_{i_1,i_2,i_3,i_4,i_5=0}^{1} C_{i_1i_2i_3i_4i_5}|i_1i_2i_3i_4i_5\rangle, \tag{31}$$

first we consider the partition $N = 5, n = 1$. In this case we have $L = 16$ and $l = 2$, so we have 2-planes in $\mathbf{C}^{16}$. The set of such 2-planes is the Grassmannian $Gr(16, 2)$. Alternatively, one can think of this space as the one parametrizing the set of lines in $\mathbf{CP}^{15}$. Now a five-qubit state is characterized by the pair of vectors $\mathbf{Z}_0$ and $\mathbf{Z}_1$ forming the 16×2 matrix $Z_{\alpha a}$ ($\alpha = 0, 1, \ldots, 15, a = 0, 1$). Now the invariant $E_1^{(5)}$ has the same form as equation (18) where now $g = \varepsilon \otimes \varepsilon \otimes \varepsilon \otimes \varepsilon$. Written out explicitly, we see that the quantities $\mathbf{Z}_0 \cdot \mathbf{Z}_0$ and $\mathbf{Z}_1 \cdot \mathbf{Z}_1$ have the same structure as the one appearing in equation (28). Indeed, it is known that the invariant H of degree 2 responsible for this structure defines a quadratic binary form in the variables x and y. The discriminant of this form defines an invariant of degree 4 [18]. This discriminant is precisely of the form (18) we are already familiar from the definition of the 3-tangle via the use of Cayley's hyperdeterminant. We can define four other invariants $E_1^{(1)}$, $E_1^{(2)}$, $E_1^{(3)}$ and $E_1^{(4)}$ similarly. One can show [18] that the invariants $E_1^{(j)}$ with $j = 1, \ldots, 5$ are algebraically independent.

Let us now consider the partition $N = 5, n = 2$. In this case, $Z_{\alpha a}$ is a 8×4 matrix. Since $N - n = 3$ is odd, g is antisymmetric; hence, $\mathbf{Z}_a \cdot \mathbf{Z}_b = -\mathbf{Z}_b \cdot \mathbf{Z}_a$. Hence, the invariant $E_2^{(45)}$ has the form

$$E_2^{(45)} = 16|\mathrm{Det}(\mathbf{Z}_a \cdot \mathbf{Z}_b)|^{1/2}. \tag{32}$$

Since the determinant of an even-dimensional antisymmetric matrix can always be written as a square (the Pfaffian), we can write this as

$$E_2^{(45)} = 16|\mathbf{Z}_0 \cdot \mathbf{Z}_1\mathbf{Z}_2 \cdot \mathbf{Z}_3 - \mathbf{Z}_0 \cdot \mathbf{Z}_2\mathbf{Z}_1 \cdot \mathbf{Z}_3 + \mathbf{Z}_0 \cdot \mathbf{Z}_3\mathbf{Z}_1 \cdot \mathbf{Z}_2|. \tag{33}$$

Note that there are ten entanglement monotones of this kind based on a partition of the form $5 = 3 \oplus 2$. However, these invariants cannot be independent from $E_1^{(j)}$ ($j = 1, \ldots, 5$) due to the results of [18] showing that the number of algebraically independent fourth-order invariants is 5.

4.4. The N-qubit invariants of Wong and Christensen

In their paper, Wong and Christensen have introduced a potential entanglement measure calling it the N-tangle [3]. In our notation, they are just the invariants $E_1^{(N)}$ based on the partition $N = N - 1 \oplus 1$ corresponding to Grassmannians $Gr(2^{N-1}, 2)$ of 2-planes in $\mathbf{C}^{2^{N-1}}$. In [3], it was observed that for N even these invariants can be written as a square of the pure state concurrence [6]. This structure is indeed exhibited by our two- and four-qubit invariants (22) and (27). This result easily follows from the observation that the matrix g of equation (20) in this case is antisymmetric. Since the pure state concurrence is a permutation invariant, we conclude that the invariants $E_1^{(N)}$ for N even are also permutation invariants. For the four-qubit case, we recover the well-known permutation invariance of H of [7].

We also see that the invariants $E_n^{(\{k\})}$ arising from the partition $N = N - n \oplus n$ can always be written as a square of another invariant when $N - n$ is odd. This again follows from the antisymmetry of g and the fact that the determinant of an even-dimensional antisymmetric matrix can be represented as a square of the Pfaffian. The simplest example of a Pfaffian is the combination (the Plücker relation) appearing in equation (33).

5. Conclusions

In this paper, we have introduced a class of N-qubit entanglement monotones based on bipartite decompositions $N = N - n \oplus n$ of the Hilbert space $\mathcal{H} \simeq \mathbf{C}^{2^N}$. This decomposition has naturally led us to the use of Grassmannians $Gr(L, l)$ of l-planes in $\mathbf{C}^L$, where $L = 2^{N-n} \geqslant l = 2^n$ is the natural structure characterizing the geometry of a subclass of N-qubit entanglement. Our construction of such monotones was based on the paper of Emary [16]. The new monotones unlike the ones in [16] are SLOCC invariants, i.e. invariant under stochastic operations and classical communication. We have shown how the well-known invariants such as the concurrence, the 3-tangle, the N-tangle and some of the four- and five-qubit invariants introduced recently can be obtained as special cases.

There are a lot of interesting possibilities left to be explored. The most important is of course to see what is the physical meaning of our monotones $E_n^{(\{k\})}$, for what kind of states we have $E_n^{(\{k\})} = 0$, etc. Moreover, an interesting development would be the extension of the approach initiated in [14] of characterizing different SLOCC classes of entanglement via studying the intersection properties of $(l-1)$-planes in $\mathbf{CP}^{L-1}$. A geometric approach of this kind would establish interesting links between the theory of entanglement and twistor theory [14, 19]. Such interesting questions will be addressed in a future publication.

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